THE OPTIMAL NUMBER OF MEMBER COUNTRIES
IN AN ECONOMIC UNION

by

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Abstract: We study the choice of optimal membership in an economic union when member-countries’ national governments set their tax policies non-cooperatively. Federal policy (in the form of union membership) has a higher constitutional status than national policies (in the form of income tax rates). This allows federal policy to reduce the inefficiencies arising from uncoordinated national policies. We show that equilibrium membership is inefficiently low and decreases with those factors that generate Nash-type inefficiencies. When these inefficiencies take the form of tax competition for mobile tax bases and free riding on other countries’ contribution to international public goods, one can rationalize small size unions only. The normative lesson is that union enlargement requires a switch from uncoordinated to coordinated national policies.

Keywords: Clubs, Capital mobility, Federalism.
JEL classification numbers: H2, H7.

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I. INTRODUCTION

The study of economic unions (formation, enlargement and stability) has built upon the theory of clubs.\(^1\) Interest in this topic reflects the changing number of countries in the world (e.g. the breakup of the former Soviet Union into many countries, and the German unification). It also reflects developments in the formation of economic unions or federations (e.g. the European Union, and NAFTA).

An interesting example of a club is the European Union (EU). The general feeling among economists is that EU enlargement is too ambitious. Economic concerns become stronger given the fact that most national (i.e. domestic) fiscal policies are still selected in a decentralized, uncoordinated way. For instance, national governments in the EU still choose their own tax policies to maximize a domestic objective. Yet, at the same time, supra-national or federal authorities design institutions and choose common policies to internalize cross-border externalities and promote integration within the union. A key federal policy is the choice of union membership. What is the “optimal” number of member-countries in an interdependent world where domestic fiscal policies are uncoordinated?

The present paper studies the choice of optimal membership in an economic union when member-countries’ national governments set their tax policies non-cooperatively. The setup is a multi-country general equilibrium model with international capital movements and public goods with cross-border externalities. Within this setup, we formalize the choice of national and federal policies. Federal policy (in the form of union membership) is chosen prior to national policies (in the form of income tax rates). In other words, federal policy has a higher constitutional status than national policies.\(^2\) This hierarchical structure allows federal policy to internalize the behavior of national governments, and hence reduce the inefficiencies arising from uncoordinated national fiscal policies.

At the root of the problem of optimal membership is the tradeoff in the size of population, as studied by the theory of clubs. In the basic model of clubs, on the one

\(^{1}\) For a review of the literature on clubs, see e.g. Cornes and Sandler (1996). For the theory of economic unions, see e.g. Bolton and Roland (1997), Alesina and Spolaore (1997), Alesina et al. (2000) and Alesina et al. (2001). Details are given in section VI below.

\(^{2}\) As Persson and Tabellini (1996, p. 635) point out, this reflects the EU system and “it captures a situation where the federation has more commitment power and its policy choices are less easily reversible than the national choices”. The same authors also say that this is one of the main differences
hand, a larger club can benefit from greater economies of scale in the provision of public goods; on the other hand, there are congestion costs when a new member joins. Then, subject to this tradeoff in the size of population, a hypothetical planner chooses club membership to maximize the utility of the representative member. The conceptual difference of our work is that we study how this basic tradeoff in the size of population, and hence the eventual choice of club membership, are affected by the way national fiscal policies are set. When federal policy chooses club membership, it takes into account that national governments have played the nationalistic Nash game in national tax policies. Equilibrium membership will therefore reflect this.

Once cross-country interaction is non-cooperative, the type of international spillover effects becomes crucial. Here, we focus on two popular cross-country spillover effects. The first one is generated by international capital mobility coupled with distortionary taxation. As is known, when national policies are non-cooperative, this can lead to tax competition for mobile tax bases. The second spillover effect is generated by the so-called international public goods, i.e. public goods whose benefits extend beyond national boundaries (e.g. security, defense, the environment, border controls, health, infrastructure, etc). As is known, when national policies are non-cooperative, this leads to free riding on other countries’ contribution to the public good. We choose these two problems (i.e. tax competition and free riding) because they are believed to be important both by the academic literature and the economic press.3

Our results are as follows. It is convenient to report first what happens for given membership. Each Nash-type inefficiency (corresponding to a particular spillover effect) ceteris paribus deteriorates with the size of population. Specifically, in our model, the problems of tax competition and free riding each gets worse as the number of countries increases.4 This rather standard result has important implications for the choice of membership.

We then endogenize membership. When a “federal authority” chooses club membership,5 it internalizes the tradeoff in the number of countries. On the one hand, a larger size benefits from larger tax bases and revenue contributions needed for the

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3 For the academic literature, see e.g. Persson and Tabellini (1992, 1995), Inman and Rubinfeld (1996), Razin and Sadka (1999), Sorensen (2000) and Devereux et al. (2002).

4 For instance, in the EU, difficulties in getting agreements in the policy areas of structural funds and CAP get worse when the number of member-countries increases.

between the “EU” system and the “US” system. In the EU, supranational and national governments are vertically ordered, while in the US they are horizontally ordered.
provision of international public goods; this is basically a scale effect from higher membership. On the other hand, in addition to standard congestion problems, the inefficiencies arising from decentralized national policies increase with club size. It is this tradeoff that will determine “equilibrium” membership. We show that equilibrium membership is lower than “efficient” membership. The former corresponds to a case in which membership is chosen subject to a Nash game in national tax policies. The latter corresponds to the benchmark, efficient case in which membership is chosen subject to cooperative national tax policies.6

Therefore, to the extent that there are inefficiencies arising from decentralized national policies, equilibrium membership is too low. Intuitively, the federal authority finds it optimal to choose a relatively small membership in order to reduce the degree of Nash-type inefficiencies that naturally increase with the size of population.7 This result does not depend on the specific type of inefficiency assumed; any Nash-type inefficiency would give the same result. Therefore, the normative conclusion is that, ceteris paribus, union enlargement requires a switch from uncoordinated to coordinated national fiscal policies.

Numerical solutions confirm our analytical results. Namely, equilibrium union membership and national income tax rates are both inefficiently low. The same solutions reveal that equilibrium membership decreases with the degree of capital mobility. Intuitively, the inefficiency associated with the race-to-the bottom for internationally mobile tax bases gets worse when capital mobility, or the size of population, increases. Hence, as capital mobility increases, the federal authority finds it optimal to set an even smaller membership to mitigate the adverse effects of tax competition. The effect of the degree of cross-border externalities delivers a similar message: as this degree increases so that the incentive to free ride on other countries increases, the gap between equilibrium and efficient membership gets wider. Thus,

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5 In our setup, the decision of the federal authority coincides with the decision that it is optimal for the individual citizen-voter (see below).

6 Thus, the equilibrium outcome is inefficient. Note that the literature on the optimal number of nations has also distinguished “optimal” from “equilibrium” outcomes [see, in particular, Alesina and Spolaore (1997) and Alesina et al. (2001)]. However, in that literature, this has a different meaning: the former refers to the social planner outcome, while the latter usually refers to a democratically chosen number of nations. Instead, in our paper, the number of nations is always chosen by a planner. Thus, the distinction between “equilibrium” and “efficient” outcomes has to do with the way national policies are set. See also section VI below.

7 Inman and Rubinfeld (1996, pp. 329-331) also argue that by limiting the number of states in a federal economy can be a constitutional way of reducing the inefficiencies that arise within a decentralized federal economy.
the general message is that union membership should decrease with any factors that exacerbate Nash-type inefficiencies.

Actually, our numerical simulations reveal that, to the extent that decentralized national policies lead to tax competition and free riding, one can rationalize small size unions only. In terms of policy implications, only if countries in Europe start cooperating their fiscal (spending-tax) policies, factors like increasing scale effects, falling congestion costs, etc, can play a role in justifying the EU and its planned enlargement. Cooperation is a precondition to enlargement.

The rest of the paper is as follows. Section II presents a world economy and solves for a competitive equilibrium. Section III solves for national policies. Section IV solves for federal policy in the form of union membership. Section V presents conclusions and policy lessons. Section VI reviews the literature.

II. THE MODEL AND WORLD COMPETITIVE EQUILIBRIUM

Consider a world economy composed of $M$ countries, indexed by $i = 1, 2, ..., M$. The number of countries, $M$, is chosen by a federal authority. Each country $i$ is populated by a representative private agent and a national government. The private agent consumes and invests at home and abroad, where investment abroad implies a mobility cost. The national government in each country is benevolent and uses income taxes to finance the provision of a public good, whose benefits extend beyond national boundaries. We assume the Source Principle of international taxation, according to which domestic and foreign investors are being taxed at the same rate (this is the principle used in most countries).

As in the basic model of clubs, we assume that a central or federal authority creates the world from scratch. Alternatively, we could assume that the world economy is composed of $N$ potential members, where $N$ is exogenous and $M \leq N$ is chosen by the central authority. In that case, one has to distinguish between members, $M$, and non-members, $(N - M)$, whenever optimal conditions are derived [see Cornes and Sandler (1996) for the general theory of clubs; see also e.g. Alesina et al. (2001) for an international union model]. We report that we have experimented with richer models like this (for instance, a model with member and non-member countries where non-members’ capital stock is a fraction of members’ capital stock and where a planner chooses $M$ to maximize the utility of member countries) and the spirit of our results is unaffected (of course, quantitative results depend on e.g. the way members and non-members differ, the assumed spillovers between members and non-members, whether only members’ utility is considered when membership is chosen or whether both members and non-members’ utility matters, etc; see Cornes and Sandler (1996)). Here, we choose to present the simplest possible model specification. However, it is important to point out that all this has to do only with what Alesina et al. (2001) call the “initial, formation stage”. There is a second stage,
II. 1 Informal description of the model and definition of equilibrium

The sequence of moves is as follows: first, a federal authority chooses club membership; in turn, national governments simultaneously choose their tax policies; finally, private agents make their consumption-investment choices. We take as given the design of this three-tier hierarchical structure (i.e. federal government, national governments and private sectors), the assigned policy responsibilities, and the sequence of moves.

We will solve the game backwards. Thus, we first solve the last stage. Private agents in each country maximize their lifetime utility by choosing consumption and investment at home and abroad. In doing so, they act competitively by taking prices and policies as given. The solution to this stage will give a World Competitive Equilibrium. This is for any feasible national fiscal policies and any number of countries.

In the second stage, we solve for Nash national tax policies and the associated level of government expenditures. Each national government chooses its own income tax rate by being concerned only about the welfare of its own citizen and by taking as given the income tax rates of other national governments. When national governments choose their tax policies, they take into account the World Competitive Equilibrium of the previous stage. On the other hand, they take the number of countries as given.

Finally, in the first stage, a federal authority chooses the number of countries (or club membership) to maximize the utility of the representative citizen. In doing so, it takes into account the outcomes of all previous stages. The solution to this problem will give equilibrium membership.

But we also need a benchmark case. As said above, this is defined to be a case in which the federal authority chooses club membership subject to fully cooperative national policies. The solution to this problem will give a Pareto-efficient membership. This case corresponds to the social planner’s optimum.

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9 See also Yi (1997) for a two-stage structure, where in the first stage players form coalitions, and in the second stage they engage in a non-cooperative game given the coalition structure. Sequential actions are also assumed in the literature on the optimal formation of nations [see e.g. Bolton and Roland (1997), Alesina and Spolaore (1997), Perroni and Scharf (2001) and Alesina et al. (2001)]. Caplan et al. (2000) study different sequential structures and their implications for federal and national policies.
The rest of this section solves the third, last stage of the game, i.e. a world competitive equilibrium for given national policies and the number of countries.\textsuperscript{10}

II. 2 Behavior of private agents

The model is a multi-country version of the model used by Persson and Tabellini (1992). We choose this model because it is simple. Consider a two-period economy, with one private and one public good, and linear $AK$ production technologies. Each country $i = 1, 2, \ldots, M$ is populated by a private agent and a national government.

The private agent in country $i$ maximizes utility:

$$U^i = U(c^i, d^i; H^i, M)$$ (1)

where $c^i$ and $d^i$ are private consumption in the first and second period respectively, $H^i$ is the international public good from the point of view of country $i$ [see also equation (5f) below], and $M$ is union membership. The utility function is increasing in $c^i$, $d^i$ and $H^i$, and decreasing in $M$. The negative affect of $M$ captures crowding, or congestion, costs.\textsuperscript{11} Equations like (1) have been widely used in the literature on clubs.\textsuperscript{12}

Without loss of generality, we assume an additively separable function of the form

$$\log c^i + d^i + \nu H^i - \frac{\lambda M^2}{2},$$

where the parameter $\nu > 0$ is the weight given to public services, and the parameter $\lambda > 0$ measures congestion costs from higher membership.\textsuperscript{13}

The first-period budget constraint of the private agent in each country $i$ is:

$$c^i + k^i + \sum_{j \neq i}^M k^j = e^i$$ (2a)

That is, private agents start with an exogenous endowment $e^i > 0$ and then choose how much to consume and how much to invest at home and abroad. Private agents in country $i$ can invest $k^i$ at home in country $i$, and $k^j$ abroad in countries $j$, where $j \neq i$.

\textsuperscript{10} If we assume away the second stage, we are back to the basic model of the literature on clubs.

\textsuperscript{11} An increase in the number of countries typically increases the complexity of communication, bargaining and representation [see e.g. Inman and Rubinfeld (1997)].

\textsuperscript{12} See e.g. Mueller (1989, pp. 152-4) and Drazen (2000, pp. 393-4). See also Alesina et al. (2000, p. 1282) in the literature on the optimal number of nations.

\textsuperscript{13} See Persson and Tabellini (1992) and Alesina et al. (2000) for similar functional forms.
The second-period budget constraint of the private agent in country \( i \) is:

\[
d^i = (1 - \theta^i) A^i k^{ii} + \sum_{j \neq i}^M (1 - \theta^j) A^j k^{ij} - \sum_{j \neq i}^M B(k^{ij}; \beta^{ij})
\]  

(2b)

where the parameter \( A^i \) is capital return in country \( i \), \( 0 < \theta^i < 1 \) is the income tax rate in \( i \), and the function \( B(k^{ij}; \beta^{ij}) \) captures net mobility costs from investing \( k^{ij} \) abroad, i.e. this function is increasing in \( k^{ij} \), while the parameter \( \beta^{ij} > 0 \) measures the size of net mobility cost from investing \( k^{ij} \) abroad [see also Persson and Tabellini (1992)].

Without loss of generality, we assume \( B(k^{ij}; \beta^{ij}) = \frac{\beta^{ij} (k^{ij})^2}{2} \).

Private agents take prices, tax policy, public services and the number of countries as given. Assuming an interior solution, the first-order conditions are (2a)-(2b) and the Euler equations:

\[
\frac{1}{c^i} = (1 - \theta^i) A^i
\]  

(3a)

\[
\frac{1}{c^i} = (1 - \theta^i) A^i - \beta^{ij} k^{ij} \quad \text{for } j \neq i \text{ and } j = 1,2,\ldots,M
\]  

(3b)

so that (3a)-(3b) imply:

\[
(1 - \theta^i) A^i = (1 - \theta^i) A^i - \beta^{ij} k^{ij}
\]  

(3c)

i.e., without uncertainty, net returns are equalized across countries via capital mobility.

**II. 3 National government’s budget constraint**

Each national government \( i \) spends \( h^i \) on the public good. This is financed by income taxes. Since we use the source principle of international taxation, the national government \( i \) taxes domestic and foreign investors at the same rate, \( 0 < \theta^i < 1 \). Thus, the budget constraint of the national government in country \( i \) is:
\[ h^i = \theta^i A^i k^i \]  

where \( k^i \equiv (k^{ii} + \sum_{j(i) \neq i}^M k^{ij}) \) is the capital stock (and the tax base) in country \( i \).

II. 4 World Competitive Equilibrium (for given policy and the number of countries)

We now solve for a World Competitive Equilibrium (WCE). In this equilibrium: (i) all private agents maximize utility; (ii) all constraints are satisfied; (iii) all markets clear. This is for given national policies (\( \theta^i \)) and federal policy (\( M^i \)).

Using (2)-(4) above, we have for each country \( i \) in a WCE:

\[ c^i = \frac{1}{(1 - \theta^i) A^i} \] (5a)

\[ d^i = (1 - \theta^i) A^i k^{ii} + \sum_{j(i) \neq i}^M (1 - \theta^j) A^j k^{ij} - \sum_{j(i) \neq i}^M \beta^{ij} k^{ij} \] (5b)

\[ h^i = \theta^i A^i \left( k^{ii} + \sum_{j(i) \neq i}^M k^{ij} \right) \] (5c)

where,

\[ k^{ii} = c^i - \frac{1}{(1 - \theta^i) A^i} - \sum_{j(i) \neq i}^M k^{ij} \] (5d)

\[ k^{ij} = \frac{(1 - \theta^i) A^j - (1 - \theta^i) A^i}{\beta^{ij}} \] (5e)

where (5a), (5b), (5c), (5d) and (5e) give respectively the first-period consumption, the second-period consumption, government expenditure on the public good, capital invested at home, and capital invested abroad. This is for each country \( i = 1, 2, \ldots, M \).\(^{14}\)

\(^{14}\) Without capital mobility costs, \( \beta^{ij} = 0 \), the WCE would be indeterminate. Specifically, in each country \( i \), we would have two equations only, while there are three endogenous variables (\( c^i, k^{ii} \) and \( k^{ij} \)). This is a known problem when factor returns are exogenous. This is why introducing mobility costs is particularly useful; it gives a well-defined solution for the WCE without complicating the model. Alternatively, Kehoe (1989) assumes that domestic capital is owned only by domestic investors (\( k^{ii} = 0 \)). Also, note that the presence of capital mobility costs allows existence of a Nash equilibrium in national tax policies (see the next section below). Without these costs, and with exogenous factor returns, there would be nothing left for national governments to choose (see (3c) above).
To close the world economy, we have to model the international public good, $H^i$, in equation (1) above. Following Alesina et al. (2001), we assume:

$$H^i \equiv h^i + \gamma \sum_{j \neq i} h^j$$

(5f)

where the parameter $0 \leq \gamma \leq 1$ measures the extent of the benefit country $i$ enjoys from other countries $j \neq i$ providing the international public good. We assume $0 < \gamma < 1$, so that country $j$’s provision of the public good has a lower effect on country $i$’s utility than $i$’s own provision of the public good. Note that a positive $\gamma$ can be also thought of as a scale effect [see e.g. Backus et al. (1992)].

We sum up this section. We have solved for a World Competitive Equilibrium (WCE). This is summarized by (5a)-(5f). These equations give a convenient closed-form analytical solution for equilibrium allocations, as functions of national tax rates, $\theta^i$, and the number of countries, $M$. In the next section III, national governments will choose tax policy, $\theta^i$. The number of countries, $M$, will be chosen in section IV.

III. DETERMINATION OF NATIONAL TAX POLICIES

We move on to the second stage of the game and endogenize national policies. National tax rates, $\theta^i$, are determined by a Nash game among national governments. When they choose $\theta^i$, benevolent national governments take into account the World Competitive Equilibrium specified above. On the other hand, they take the number of countries, $M$, as given. For simplicity, we will focus on symmetric equilibria in national policies. That is, ex post, $\theta^i = \theta^j \equiv \theta$, $c^i = c^j \equiv c$, $d^i = d^j \equiv d$, $k^i = k^j \equiv k$, etc, where $i \neq j$. Focusing on symmetric equilibria is not restrictive for what we want to do, which is to study how incentives affect the choice of club membership, $M$.

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15 See also Backus et al. (1992) and Alesina and Wacziarg (1999) for similar modeling, although the cross-country externalities are on the production side. The main results do not depend on this.

16 As in Alesina et al. (2001), if $\gamma = 0$, the public good is national; if $\gamma = 1$, cross-border spillover effects are perfect.

17 In a symmetric equilibrium, there are no capital flows across countries ex post [see (5e)]; this is as in Persson and Tabellini (1992).
III. 1 Uncoordinated (Nash) national tax policies

Each national government \(i\) chooses \(\theta^i\) to maximize (1) subject to (5a)-(5f) above. In doing so, it takes \(\theta^j\), where \(j \neq i\), as well as \(M\) as given. Using (5a)-(5f) into (1), deriving the first-order condition for \(\theta^i\), invoking symmetry, and assuming existence of an interior solution, we obtain (from now on, we omit country-superscripts):

\[
A\left(e - \frac{1}{(1-\theta)A}\right) = \nu \left[A\left(e - \frac{1}{(1-\theta)A}\right) - \frac{\theta}{(1-\theta)^2} - \frac{2\theta(A-1)(1-\gamma)}{\beta}\right]
\]

which is an equation in \(\theta\) only. Then, (6) implies:

**Result 1:** Given the number of countries, \(M\): (i) A Symmetric Nash Equilibrium (SNE) in national policies is summarized by the tax rate that solves equation (6). (ii) This Nash tax rate, denoted as \(0 < \tilde{\theta} < 1\), is unique. Also, \(\tilde{\theta} = \theta(M; \beta, \gamma, \nu, e)\).

*Proof: See Appendix.*

Notice that the Nash tax rate decreases with the number of countries, \(M\), and increases with capital mobility costs, \(\beta\). As we discuss in subsection III.3 below, these are intuitive results.

III. 2 Coordinated national tax policies

It is useful for what follows in the next section to solve also for the case in which national tax policies are chosen cooperatively. Now, a hypothetical planner chooses all \(\theta^i\) to maximize the sum of (1) over all countries by taking \(M\) as given.

Working as above, it is straightforward to show that in Symmetric Cooperative Equilibrium (SCE), we have instead of (6):

\[
A\left(e - \frac{1}{(1-\theta)A}\right) = \nu[1 + \gamma(M - 1)] \left[A\left(e - \frac{1}{(1-\theta)A}\right) - \frac{\theta}{(1-\theta)^2}\right]
\]

(7)
which is an equation in $\theta$ only. Then, (7) implies:

**Result 2:** Given the number of countries, $M$ : (i) A Symmetric Cooperative Equilibrium (SCE) in national policies is summarized by the tax rate that solves equation (7). (ii) This cooperative tax rate, denoted as $0 < \theta^* < 1$, is unique. Also, $\theta^* = \theta(M; \gamma, \nu, e)$.

*Proof:* See Appendix.

Notice that the cooperative tax rate is increasing in the number of countries, $M$, and is unaffected by capital mobility costs, $\beta$. That is, the comparative static properties in Result 2 differ from those in Result 1. Results are explained in the next subsection.

III. 3 Properties of national tax policies

Inspection of equations (6) and (7) reveals:

**Proposition 1:** Given the number of countries, $M$ : (i) Since there are cross-country spillover effects, there are typical inefficiencies when national fiscal policies are uncoordinated (Nash). (ii) Ceteris paribus, each Nash-type inefficiency (corresponding to a particular spillover effect) deteriorates with the size of population, $M$. (iii) Since, in this model, all spillover effects generate positive policy externalities and hence pull the Nash strategies in the same direction, Nash strategies are inefficiently low, $0 < \bar{\theta} < \theta^* < 1$, and the overall inefficiency increases monotonically with $M$.

These are standard properties. However, they will have interesting implications for the choice of $M$ in the next section. What is important to our final results will be properties (i) and (ii).

The rest of this subsection explains the above proposition, in case it is not clear. We start with properties (i) and (ii). Recall that there are two spillover effects: international capital movements and international public goods. Irrespective of whether these spillovers generate positive or negative externalities (see next
paragraph), each Nash-type inefficiency (associated with a particular spillover) deteriorates ceteris paribus with the size of population.\footnote{Observe the last term on the right-hand side of (6), \( \frac{2\Delta^2 \theta(M-1)(1-\gamma)}{\beta} \), which is absent from (7).}

Property (iii) arises simply because both spillover effects work in the same direction generating a positive policy externality. A positive externality means that there is a positive external welfare effect from country \( j \)'s tax rate to country \( i \)'s welfare, where \( i \neq j \). Specifically, in the case of international capital movements, an increase in the foreign tax rate leads to capital flight, and hence higher economic growth in the domestic country. In the case of international public goods, an increase in the foreign tax rate leads to higher tax revenues abroad, and hence higher contribution to the provision of international public goods, which again increases domestic welfare. Then, since all policy externalities are positive: (a) The Nash tax rate is unambiguously lower than the cooperative tax rate; i.e. \( 0 < \bar{\theta} < \hat{\theta} < 1 \) (see proof of Result 2 in the Appendix). (b) The Nash tax rate decreases monotonically with the size of population, \( M \).\footnote{Our results are consistent with general results for symmetric equilibria. Specifically: (a) In the presence of positive (resp. negative) externalities, players’ strategies are inefficiently low (resp. high) in a Nash equilibrium relative to a cooperative one. See e.g. Cooper and John (1988). (b) Ceteris paribus, each Nash-type inefficiency gets worse with the size of population. Thus, when a Nash strategy is inefficiently low (resp. high), it decreases (resp. increases) with the size of population. See e.g. Kehoe (1987) and Philippopoulos and Economides (2003). Note that the relation between the size of population and Nash strategy is unambiguous because we have solved for symmetric equilibria. In cases with asymmetric equilibria, results are ambiguous [for a survey, see Myles (1995, pp. 284-7)].} Notice that, by contrast, the cooperative tax rate, \( \hat{\theta} \), increases with \( M \). This happens because cooperative policies are not affected by Nash-type inefficiencies and, at the same time, internalize the positive, scale effects from public good provision [see (5f) above]. All this means that, while the Nash tax rate is decreasing in \( M \), the cooperative tax rate is increasing in \( M \) [see also Alesina and Wacziarg (1999, p.20)].\footnote{For analogous reasons, \( \bar{\theta} \) is increasing in capital mobility costs \( \beta \) (see Result 1), while \( \hat{\theta} \) is independent of \( \beta \) (see Result 2). Intuitively, the smaller is \( \beta \), the higher is the elasticity of capital movements with respect to tax rate differentials, and the fiercer is the competition for mobile tax bases through tax cuts. This is a standard result in the literature on tax competition [see e.g. Persson and Tabellini (1995)]. By contrast, cooperative policies are not affected by such Nash-type inefficiencies, so that \( \hat{\theta} \) is independent of \( \beta \). Also, notice that when \( \gamma = 1 \), i.e. in the limiting case in which...} It also means that tax competition and free-riding pull the
Nash tax rate, $\tilde{\theta}$, in the same direction, so that the degree of overall inefficiency, arising from decentralized national policies, unambiguously increases as $M$ increases.21

III. 4 Summarizing results
At this stage, it is useful to organize algebraic results. We have solved for a Symmetric Nash Equilibrium (SNE) and a Symmetric Cooperative Equilibrium (SCE) in national policies. Specifically, in a SNE, the tax rate is given by (6); in a SCE, the tax rate is given by (7). In turn, (5a)-(5f) imply the resource allocation:

\[ c = \frac{1}{(1 - \theta)A} \] \hspace{1cm} (8a)

\[ d = (1 - \theta)A \left( e - \frac{1}{(1 - \theta)A} \right) \] \hspace{1cm} (8b)

\[ H = [1 + \gamma(M - 1)]A \left( e - \frac{1}{(1 - \theta)A} \right) \theta \] \hspace{1cm} (8c)

Therefore, equations (6) and (8a)-(8c) characterize a Symmetric Nash Equilibrium (SNE) in national policies, while equations (7) and (8a)-(8c) characterize a Symmetric Cooperative Equilibrium (SCE) in national policies. This is for any number of countries, $M$. The next section will endogenize $M$.

IV. DETERMINATION OF CLUB MEMBERSHIP

We now turn to the first stage of the game, in which a federal authority chooses $M$ to maximize the utility of the representative citizen (since we solve for a symmetric equilibrium, this coincides with the decision that it is optimal for the individual citizen-voter).22 In doing so, the federal authority takes into account all previous stages. As said above, we will distinguish two cases: the equilibrium case in which international spillovers from public good provision are perfect [see (5f)], the Nash-type inefficiency due to tax competition vanishes in (6). However, this does not mean that Nash and cooperative policies coincide. Even when $\gamma = 1$, there are still inefficiencies due to free riding [compare (6) and (7)].

21 If $\beta \rightarrow \infty$ (i.e. there are huge mobility costs so that capital mobility is practically impossible and tax competition does not arise) and $\gamma = 0$ (i.e. the public good is only local so that there is no free riding behavior), Nash-type inefficiencies vanish. See (6) and (7) above.

22 See also e.g. Park and Philippopoulos (2003).
is chosen subject to Nash national tax policies, and the Pareto-efficient case in which \( M \) is chosen subject to coordinated national tax policies; where the latter case will serve as a benchmark.

**IV. 1 Equilibrium membership**

The federal authority chooses \( M \) to maximize (1) subject to the Symmetric Nash Equilibrium in national tax policies. Specifically, it chooses \( M \) to maximize

\[
\log c + d + \nu H - \frac{\lambda (M^2)}{2}
\]

where \( c, d \) and \( H \) are given by (8a), (8b) and (8c) respectively, and the tax rate \( \tilde{\theta} \) is given by (6). The first-order condition gives:

\[
v\gamma \tilde{\theta} A \left( e - \frac{1}{(1 - \tilde{\theta})A} \right) = \lambda M - \nu (M - 1) \left[ \frac{2A^2 \tilde{\theta} (1 - \gamma)}{\beta} + \gamma \left( A \left( e - \frac{1}{(1 - \tilde{\theta})A} \right) - \frac{\tilde{\theta}}{(1 - \tilde{\theta})^2} \right) \right] \frac{\partial \tilde{\theta}}{\partial M}
\]

which says that the marginal benefit of \( M \) (the left-hand side of (9)) equals the marginal cost of \( M \) (the right-hand side of (9)).

In our model, the marginal benefit is basically a scale effect: all member-countries benefit from a larger union since this increases the tax base and contributions to the provision of the international public good.\(^{23}\) The marginal cost consists of crowding problems \( (\lambda M) \), plus efficiency costs due to non-cooperative policymaking at national level (see the second term on the right-hand side and recall that \( \frac{\partial \tilde{\theta}}{\partial M} < 0 \) in Result 1 above).

Equations (6) and (9) are two equations in \( \tilde{\theta} \) and \( M \). Let us denote their solution as \( \tilde{\theta} \) and \( \tilde{M} \). In other words, (6) and (9), jointly with (8a)-(8c), characterize a world equilibrium in which the federal authority chooses union membership optimally, given that national governments have chosen their fiscal policies non-cooperatively.

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\(^{23}\) Recall from (5d)-(5e) that \( k = e - \frac{1}{(1 - \theta)A} \) is the tax base in each country in a symmetric equilibrium. If \( \gamma = 0 \), i.e. the services of the public good do not extend beyond national boundaries or, equivalently, there are no scale effects, the marginal benefit is zero.
Now, the federal authority chooses $M$ to maximize (1) subject to the Symmetric Cooperative Equilibrium in national tax policies. Specifically, it chooses $M$ to maximize $\log c + d + vH - [\lambda(M)^2]/2$, where $c, d$ and $H$ are given by (8a), (8b) and (8c) respectively, and the tax rate $\theta^*$ is given by (7). The first-order condition gives:

$$v^* \theta^* A \left( e - \frac{1}{(1 - \theta^*) A} \right) = \lambda M \quad (10)$$

Inspection of (9) and (10) reveals that in the latter we do not have the second positive term on the right-hand side. This is because in (10) there are no Nash-type problems as in (9).

Equations (7) and (10) are two equations in $\theta$ and $M$. Let us denote their solution as $\theta^*$ and $M^*$. In other words, (7) and (10), jointly with (8a)-(8c), characterize a world equilibrium in which the federal authority chooses club membership optimally, given that national governments have chosen their fiscal policies cooperatively. Note that this solution coincides with the solution without national governments; i.e. the case in which a world central planner chooses both national policies and club membership. That is, the case in which we choose $\theta$ and $M$ simultaneously to maximize the representative member’s utility (1) subject to the world competitive equilibrium (8a)-(8c). This is the socially optimum.

It remains to compare the equilibrium solution $(\tilde{\theta}, \tilde{M})$ given by (6) and (9) with the benchmark, efficient solution $(\theta^*, M^*)$ given by (7) and (10). Details are in the Appendix. Here, we summarize results in the following proposition:

**Proposition 2:** In equilibrium, both union membership and national income tax rates are inefficiently low relative to the benchmark case in which union membership is chosen subject to cooperative national policies. Thus, $\tilde{M} < M^*$ and $\tilde{\theta} < \theta^*$.

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24 This is as in the basic model of clubs. That is, subject to the tradeoff between scale effects and congestion costs, a planner chooses the club size and the quantity of the public good to maximize the representative member’s utility.
Proof: See Appendix.

Intuitively, when national policies are decentralized, there are inefficiencies that increase with the size of population (see Proposition 1 above). Hence, when the federal authority chooses club membership, it finds it optimal to choose a relatively small membership so as to reduce the magnitude of Nash-type inefficiencies.

IV. 4 Numerical solutions

Numerical solutions confirm our analytical results. Consider, for instance, the baseline parameter values $\gamma = 0.5$, $\lambda = 0.5$, $\nu = 1.2$, $\beta = 0.2$, $A = 2$, $e = 10$. Then, as reported in Table 1, the system (6) and (9) gives $\tilde{\theta} = 0.390$ and $\tilde{M} = 1.256$ for the equilibrium case, while the system (7) and (10) gives $\theta^* = 0.766$ and $M^* = 14.456$ for the efficient reference case. Tables 1 and 2 show that in all cases $\tilde{M} < M^*$.

Tables 1 and 2 here

We focus on the effects of the two key parameters, $\gamma$ and $\beta$ (reported in Tables 1 and 2 respectively). Recall that as $\gamma$ rises, the cross-border effect from the provision of the international public good becomes larger, so that the incentive to free ride on other countries’ contribution increases. Also recall that as $\beta$ falls, transaction costs fall, so that the incentive to compete for mobile tax bases increases.

In Table 1, as $\gamma$ rises, union membership increases monotonically in both the equilibrium solution and the efficient case, but the increase in the latter is much bigger. Actually, as $\gamma$ rises, the ratio $M^*/\tilde{M}$ increases (at least up to $\gamma = 0.6$, after which the mechanical scale effects dominate the strategic free riding effects). In other words, as the free riding problem gets more acute, the gap between the efficient and equilibrium membership gets bigger.

In Table 2, as $\beta$ falls, the efficient solution remains unaffected (as shown in (7) above), while equilibrium union membership, $\tilde{M}$, decreases. As a result, as $\beta$

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25 We report economically admissible solutions only. For instance, we do not report solutions for the tax rate higher than one, or solutions for membership less than one.
falls, $\tilde{M}$ shifts further away from $M^*$ or equivalently $M^*/\tilde{M}$ increases monotonically. In other words, as the inefficiency associated with tax competition gets worse, the federal authority finds it optimal to choose an even smaller membership to mitigate the adverse consequences.

Finally, notice that the solution of equilibrium membership, $\tilde{M}$, is relatively small in all cases (this is robust to changes in parameter values). Thus, the precondition for union creation, not to mention union enlargement, is coordination of national fiscal policies.

V. CONCLUSIONS AND POLICY IMPLICATIONS

We studied the endogenous determination of union membership, when national fiscal policies are uncoordinated. The setup was a multi-country general equilibrium model with international capital movements and public goods with cross-border externalities. In choosing club membership, the federal authority, or equivalently the individual voter, had the ability to internalize externalities and reduce the inefficiencies arising from uncoordinated national fiscal policies. We showed that equilibrium membership decreases with any factors that generate or exacerbate Nash-type inefficiencies. In the particular case in which these inefficiencies take the form of tax competition and free riding, one can rationalize the formation of small size unions only.

The normative lesson is that the precondition for a bigger EU is intergovernmental cooperation. Cooperation at supranational or federal level is not enough to rationalize the enlargement process. The same is true for factors like increasing scale effects, falling congestion costs, etc. These factors can play a role in justifying a bigger EU, only if countries in Europe start cooperating their tax policies.

We believe that our results imply “a policy trilemma”: high degrees of international economic integration (here, in the form of international capital movements and international public goods), national autonomy (here, in the form of decentralized national tax policies) and union enlargement (here, in the form of an increase in union membership) are difficult to co-exist. One has to reduce the degree of one of these three things. These results are consistent with Rodrik’s (2000) argument that economic integration, nation-state policies and mass politics cannot coexist. They also resemble the well-known monetary policy trilemma: perfect capital mobility, independent monetary policy and fixed exchange rates cannot co-exist.
VI. COMPARISON WITH THE LITERATURE

Our work is directly related to two strands of literature. First, there is the literature on fiscal federalism, which has shown that, whenever cross-country spillovers exist and policies are chosen independently by each country, centralized mechanisms should be designed to correct Nash-type inefficiencies. In principle, federal or supranational policy is one of them [see e.g. Persson and Tabellini (1995), Inman and Rubinfeld (1996) and Oates (1999) for surveys]. In our paper, the federal policy instrument is the size of club.

Second, there is the theory of clubs. In the basic model of clubs, as said in the Introduction above, there is a tradeoff between economies of scale in the provision of public goods against congestion costs; subject to this tradeoff in the size of population, a planner chooses the optimal membership to maximize the utility of representative club member. The recent literature on the optimal number of nations has studied mixed clubs [see e.g. Bolton and Roland (1997), Alesina and Spolaore (1997), Alesina et al. (2000), Perroni and Scharf (2001), Casella (2001), Alesina et al. (2001); see Drazen (2000, chapter 14) for a survey of this literature]. In this literature, the key tradeoff in the size of population is between internalization of spillovers and heterogeneity in larger populations. Specifically, a larger population implies lower per capita costs of excludable public goods, but a larger distance between individual preferences and the group’s choice of the public good. In turn, this tradeoff determines the optimal number of nations.26

Here, we built on the basic model of clubs to study how the determination of union formation is also affected by members’ (cooperative or non-cooperative)

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26 In Bolton and Roland (1997), the key tradeoff is between the cost of separation and the benefit from separation due to the fact that independent decision-making yields policies closer to the preferences of national voters. If there is also capital mobility, there is an additional cost in the case of separation due to tax competition. The question is whether two pre-existing regions will choose to unite or separate. In Alesina and Spolaore (1997), the key tradeoff in the size of nations is between the falling costs of the public good provided and the fact that, as a nation gets larger, the “distance” of people from their government increases. Now there are no pre-existing nations, and the question is the optimal number of nations (like in our paper). In Alesina et al. (2000), there is a tradeoff between the higher cost associated with an increase in the size and number of countries and the gains from more available resources. In Alesina et al. (2001), the tradeoff is between taking decision in common and heterogeneous preferences. They model union formation in two stages (first creation and then enlargement). Other related papers include: In Perroni and Scharf (2001), the tradeoff is between economies of scale in public good provision and the need to provide public goods tailored to different tastes, and how this tradeoff is affected by tax competition. In Casella (2001), a larger coalition implies
behavior in general, and the way national tax policies are set in particular. That is, the emphasis has been on the implications of uncoordinated actions among the members.

As Cornes and Sandler (1996, p. 355-6) point out, the early literature on clubs was not always clear as to whether an equilibrium, or an optimum, club membership was analyzed. For instance, in the basic model of clubs, all decisions are modeled as a cooperative action, so that the outcome is a Pareto optimum for the members. A notable exception in the early literature, that did study uncoordinated behavior on the part of members and hence non-optimum membership, was Scotchmer (1985).27

The recent literature on the optimal number of nations has studied non-cooperative game representations of club formation and has distinguished between optimal and equilibrium membership (see also above). However, in most of this literature, club membership has been determined residually by, for instance, nonnegative profit conditions [see e.g. Scotchmer (1985)], incentive participation constraints [see e.g. Alesina et al. (2001)], or individuals arranging themselves into jurisdictions [see e.g. Perroni and Scharf (2001)]. In our paper, by contrast, membership is chosen optimally at federal level. This is important because our hierarchical structure allows federal policy to internalize the behavior of national governments and hence reduce the inefficiencies arising from uncoordinated national policies. This is then reflected into equilibrium membership.

Therefore, the main difference of our work is that we study how Nash inefficiencies, arising from decentralized national fiscal policies, affect the tradeoff in the size of population and how this affects the choice of club membership. We also addressed these issues in a general equilibrium model with international capital mobility and international public goods.

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27 For a survey of this part of the early literature, see Cornes and Sandler (1996, chapter 14.4).
APPENDIX

A. Result 1

Consider (6) in the text. Define the left-hand side, \( LHS = A \left( e - \frac{1}{(1 - \theta)A} \right) > 0 \), and the right-hand side, \( RHS = v \left[ A \left( e - \frac{1}{(1 - \theta)A} \right) - \frac{\theta}{(1 - \theta)^2} - \frac{2A^2(1 - \gamma)}{\beta} \right] > 0 \), where \( LHS = RHS \). Then, \( LHS_\theta = -\frac{1}{(1 - \theta)^2} < 0 \) and \( RHS_\theta = -v \left[ \frac{1}{(1 - \theta)^2} + \frac{1 + \theta}{(1 - \theta)^3} + \frac{2A^2(1 - \gamma)}{\beta} \right] < 0 \). Also, for \( 0 < \theta < 1 \), \(|RHS_\theta| > |LHS_\theta|\) from the second-order condition of the maximization problem. Hence, assuming existence of a \( 0 < \theta < 1 \),\(^{28}\) there is a unique solution for the Nash tax rate, \( \tilde{\theta} \), as shown in Figure 1 below. In turn, total differentiation in (6) implies \( \frac{\partial \tilde{\theta}}{\partial M} = -\frac{v2A^2\theta(1 - \gamma)}{\beta(LHS_\theta - RHS_\theta)} \), which is negative because \( LHS_\theta - RHS_\theta > 0 \). The other comparative static results follow easily.

Figure 1 here

B. Result 2

Consider (7) in the text. Define the left-hand side, \( LHS = A \left( e - \frac{1}{(1 - \theta)A} \right) > 0 \), and the right-hand side, \( RHS = v[1 + \gamma(M - 1)] \left[ A \left( e - \frac{1}{(1 - \theta)A} \right) - \frac{\theta}{(1 - \theta)^2} \right] > 0 \), where \( LHS = RHS \). Then, \( LHS_\theta = -\frac{1}{(1 - \theta)^2} < 0 \) and \( RHS_\theta = -v[1 + \gamma(M - 1)] \left[ \frac{1}{(1 - \theta)^2} + \frac{1 + \theta}{(1 - \theta)^3} \right] < 0 \). Also, for \( 0 < \theta < 1 \), \(|RHS_\theta| > |LHS_\theta|\) from the second-order condition of the maximization problem.

\(^{28}\) In particular, existence requires that the parameter values satisfy that for \( \theta = 0 \), \( RHS > LHS \); and for \( \theta \to 1 \), \( RHS < LHS \).
Hence, assuming existence of a $0 < \theta < 1$, there is a unique solution for the cooperative tax rate, $\theta^*$, as shown in Figure 1 above. The same figure also shows that $0 < \bar{\theta} < \theta^* < 1$; this happens because the RHS in the cooperative case is always larger than the RHS in the Nash case, while the LHS is the same in both cases. In turn, total differentiation in (7) implies

$$\frac{\partial \theta^*}{\partial M} = \frac{\nu \left( \frac{e}{A} - \frac{1}{(1-\theta)A} - \frac{\theta}{(1-\theta)^2} \right)}{(LHS_{\theta} - RHS_{\theta})},$$

which is positive because

$$\left[ A \left( e - \frac{\theta}{(1-\theta)^3} \right) - \frac{\theta}{(1-\theta)^2} \right] > 0 \text{ and } LHS_{\theta} - RHS_{\theta} > 0.$$  

The other comparative static results follow easily.

C. Proposition 2

We will solve equations (6) and (9) for $\bar{\theta}$ and $\bar{M}$, and equations (7) and (10) for $\theta^*$ and $M^*$. It is convenient to start by comparing (6) and (7). When $M = 1$, (6) and (7) coincide so that $\bar{\theta} = \theta^*$ (note that, when $M = 1$, we have to set $\nu > 1$ for a solution to exist). Then, for $M > 1$, $\theta$ and $M$ move in opposite directions along (6) (see Result 1), while $\theta$ and $M$ move in the same direction along (7) (see Result 2). This is shown in Figure 2a below.

Figure 2a here

Compare now (9) and (10). When $M = 1$, (9) and (10) coincide. Then, for $M > 1$, $\theta$ and $M$ move in the same direction along (10); this is because

$$\left[ A \left( e - \frac{1}{(1-\theta)A} \right) - \frac{\theta}{(1-\theta)^2} \right] > 0.$$  

On the other hand, (9) lies to the left of (10) as shown in Figure 2b below; this is because, for any given $M$, $\theta^* > \bar{\theta}$.

Figure 2b here

We will now combine results to get solutions for $(\bar{\theta}, \bar{M})$ and $(\theta^*, M^*)$. If these solutions exist, they are shown in Figure 2c below.
In figure 2c, the intersection(s) of (7) and (10), i.e. point $C$, lies above and to the right of the intersection(s) of (6) and (9), i.e. point $N$. This is for the following reason: As shown in figure 2c, if there is an equilibrium (i.e. if (6) and (9) on the one hand, and (7) and (10) on the other hand, intersect), $\theta^*$ should lie to the right of $\tilde{\theta}$. The question is what is the relation between $M^*$ and $\tilde{M}$. Now notice that the marginal benefit of $M$ (see the left-hand sides of (9) and (10)) is increasing in $\theta$; specifically, differentiating the marginal benefit with respect to $\theta$, we get:

$$v\gamma \left[ A \left( e - \frac{1}{(1-\theta)A} \right) - \frac{\theta}{(1-\theta)^2} \right],$$

which is positive in both the non-cooperative and cooperative case. Therefore, since $\theta^* > \tilde{\theta}$, the left-hand side of (10) is higher than the left-hand side of (9). But then the right-hand side of (10) is also higher than the right-hand side of (9). In other words, $\lambda M^* > \lambda \tilde{M} + a \text{ positive term}$. This can hold only if $M^* > \tilde{M}$. Therefore, $\theta^* > \tilde{\theta}$ and $M^* > \tilde{M}$. 


REFERENCES


**Table 1: Effects of $\gamma$**

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Notes: $\lambda=0.5$, $\nu=1.2$, $\beta=0.2$, $\Lambda=2$, $e=10$.

**Table 2: Effects of $\beta$**

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Notes: $\lambda=0.5$, $\nu=1.2$, $\gamma=0.5$, $\Lambda=2$, $e=10$. 
Figure 1

\[ \theta \sim \theta^* \]
Figure 2a

Figure 2b

Figure 2c